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Explicitly exact solutions for waves in a family of nonlinear media

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Exact wave solutions are found for the electromagnetic field inside a family of strongly nonlinear dielectric media, in which the amplitude of the displacement field is a power of the external electric field. The time period of oscillation $1/\omega$, is shown to be linear in the spatial period $1/k$, which allows for a particular standing wave solution in a finite system. The velocity of propagation along trajectories of constant field in space–time coordinates is found exactly and is proportional to a power of the amplitude of the local field. This leads to shock-wave-like solutions.

A nonlinear response of media under the application of an external field is abundant in nature, and usually occurs when the applied external field is sufficiently strong. In a strong nonlinear medium the nonlinearity appears as the leading mode of behaviour, to distinguish from weakly nonlinear materials, where the nonlinearity is a small correction to a predominant linear response. The latter case enjoys many analytic and numerical studies in the literature. In contrast, the response of strongly nonlinear systems is much less investigated, and exact results are notoriously hard to come by. Here I consider dielectric systems that follow:

$$\boldsymbol{D}(\boldsymbol{r}, t) = \varepsilon|\boldsymbol{E}(\boldsymbol{r}, t)/e|^{\beta}\, \boldsymbol{E}(\boldsymbol{r}, t)\,, \tag{1a}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \mu\, \boldsymbol{H}(\boldsymbol{r}, t)\,, \tag{1b}$$

where $\varepsilon, \mu, \boldsymbol{E}(\boldsymbol{r}, t), \boldsymbol{B}(\boldsymbol{r}, t), \boldsymbol{D}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$ are, respectively, the dielectric permittivity, the magnetic permeability, the electric field, the magnetic inductance, the displacement field and the magnetic field. The rescaling scalar e has the same units as the electric field and is introduced to take care of the dimensions. Its magnitude should be determined by microscopic mechanisms, and it will be assumed unity and consequently omitted in the following for simplicity. The power law relation between the field and the response has been found to yield to exact analysis, and was therefore employed to study static properties of nonlinear conducting networks [1] and both homogeneous and disordered dielectrics [2,3]. A unique solution to Maxwell's equations has been shown to exist only for $\beta > -1$ [2], while $\beta < -1$ many metastable solutions were discovered [4].

Concentrating on the regime $\beta > -1$, I discuss here the time dependent field within such media. I show that Maxwell's equation admit oscillatory solutions for $\boldsymbol{E}$ and $\boldsymbol{H}$, and their explicit forms are derived. I find the temporal and spatial frequencies of the resultant waves and show that they are related *linearly*. I discuss the energy flow and give an explicit form for the energy density and Poynting's vector. Finally the velocity of propagation of a signal is addressed. It is shown that the characteristic lines in such a medium are not straight, but for oscillatory solutions may rather oscillate around the straight line in space–time coordinates.

Consider a semi-infinite space $-\infty < x < \infty$, $-\infty < y < \infty$ and $0 < z < \infty$, occupied by a charge-neutral, nonconducting and nonlinear dielectric

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material, that satisfies the constitutive relations (1). A monochromatic electromagnetic wave, propagating in the positive z-direction is incident on the x–y plane at $z = 0^-$. My first aim is to find the coordinate- and time-dependence of the response that develops inside the nonlinear medium, which satisfies Maxwell's relations

$$\nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}/c , \tag{2a}$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}/c . \tag{2b}$$

Without loss of generality one can align the x and y axes in the directions of the orthogonal incident fields, such that $\boldsymbol{E}_i(z=0^-, t) = E_i(0^-, t)\, \boldsymbol{x}$ and $\boldsymbol{B}_i(z=0^-, t) = B_i(0^-, t)\, \boldsymbol{y}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are unit vectors in the x and y directions. The boundary conditions at $z = 0$ consist of continuity of the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$, so

$$E(0, t) = E_i(0, t) + E_r(0, t) ,$$

$$H(0, t) = H_i(0, t) + H_r(0, t) ,$$

where the subscript r stands for the reflected wave. For $z < 0$ it is a textbook exercise to show that

$$H_i(0^-, t) = (\varepsilon_0/\mu)^{1/2} E_i(0^-, t) ,$$

$$H_r(0^-, t) = -(\varepsilon_0/\mu)^{1/2} E_r(0^-, t) .$$

So at $z = 0^+$ we have for the magnitude of the fields

$$\sqrt{(\varepsilon_0/\mu)}\, E(0^+, t) + H(0^+, t) = \sqrt{(\varepsilon_0/\mu)}\, E_i(0^-, t) + H_i(0^-, t) .$$

The above defines unambiguously how the fields should match at $z = 0$. The matching of the transmitted wave at a boundary $z = W > 0$ for a finite medium is similarly simple.

From Maxwell's relations it is easy to see that the nonlinearity in (1) does not disturb the orientations of $\boldsymbol{E}$ and $\boldsymbol{B}$, and these fields must remain in their original directions and mutually perpendicular within the nonlinear medium for $z > 0$. The directions of the response fields $\boldsymbol{D}$ and $\boldsymbol{H}$ follow those of $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively. One can now combine eqs. (2) and use (1) to eliminate the magnetic field, which gives an equation for the magnitude of $\boldsymbol{E}$:

$$\partial_{zz} E = \nu_0^{-2}\, \partial_{tt}(|E|^{\beta} E) , \quad \text{where } \nu_0 = c/\sqrt{\mu\varepsilon} . \tag{3}$$

To find a special solution for (3) let us assume that $E(z, t)$ is separable into a product of a spatial and a temporal independent functions that can be written in the form

$$E(z, t) = R(z)\, T(t)^{1/(\beta+1)} . \tag{4}$$

This separation yields a class of solutions that is of importance for finite systems, and also gives insight into the nature of a propagating signal discussed below. Substituting (4) into (3) and dividing by $(R^{\beta+1} T^{1/(\beta+1)})$ yields two independent equations for R and T:

$$\partial_{zz} R = -K |R|^{\beta} R , \tag{5a}$$

$$\partial_{tt} T = -K \nu_0^2 |T|^{-\beta/(\beta+1)} T , \tag{5b}$$

where K is a constant which relates the two functions. Here K will be assumed positive, but it is easy to convince oneself that for $K < 0$ eqs. (5) give decreasing power law solutions. These are analogous to the traditional exponential decay in the linear case. This issue is discussed elsewhere in more detail [5].

The spatial function R can be solved for by quadratures: multiply both sides by $\partial_z R$ and integrate to obtain

$$(\partial_z R)^2 + A |R|^{\beta+2} = u_z , \tag{6}$$

where u_z is a constant of integration and $A \equiv 2K/(\beta + 2)$. Eq. (6) can be regarded as an energetic relation describing a non-dissipative motion of a particle in a potential well. The first term on the LHS of (6) resembles a kinetic energy and the second – a potential term. For $\beta > 0$ ($-1 < \beta < 0$) the potential well is steeper (shallower) than the parabolic form of the familiar harmonic oscillator (which corresponds to $\beta = 0$). So (6) must accommodate oscillatory solutions. First note that when the kinetic term vanishes, the potential term is u_z, which immediately gives the amplitude of R,

$$a_R = (u_z/A)^{1/(\beta+2)} . \tag{7}$$

Correspondingly, the amplitude of $\partial_z R$ is $u_z^{1/2}$. Eq. (6) can be solved for z in the form of an indefinite integral over R:

$$u_z^{1/2}(z - z_0) = \int^{R} \frac{\mathrm{d}R'}{\pm[1 - (A/u_z)|R'|^{\beta+2}]^{1/2}} , \tag{8}$$

where z_0 is determined by the boundary conditions. Changing variables to $\zeta = (A/u_z)|R|^{\beta+2}$ ones gets

$$(A/u_z)^{1/(\beta+2)} u_z^{1/2}(z - z_0) = \frac{\pm 1}{\beta+2} \mathscr{B}_\zeta\left[\frac{1}{\beta+2}, \frac{1}{2}\right] , \tag{9}$$

where $\mathscr{B}_\zeta(a, b)$ $(0 \le \zeta \le 1)$ is the incomplete beta function [7]. By expanding the integrand in (8) it can be seen that R is linear in $(z - z_0)$ near $z - z_0 = 0$, while near the maximum in R, $R_{\max} - R \sim |z - z_{\max}|^2$. These behaviours are independent of β and generalize the linear case, when R is sinusoidal. $R(z)$ is shown in fig. 1 for $\beta = -\frac{1}{2}$ and 1, in the first quarter of the period. The field in the rest of the period is a mirror image of that shown and is facilitated by the occurrence of both signs in the integral (8), combined with the possibility of both signs for R' in the denominator in (8). The period of oscillation is $[4/(\beta+2)]\mathscr{B}(1/(\beta+2), \frac{1}{2})$, where $\mathscr{B}(a, b) = \mathscr{B}_{\zeta=1}(a, b)$ is the usual beta function [6]. This period is the analog of the 2π for $\beta = 0$.

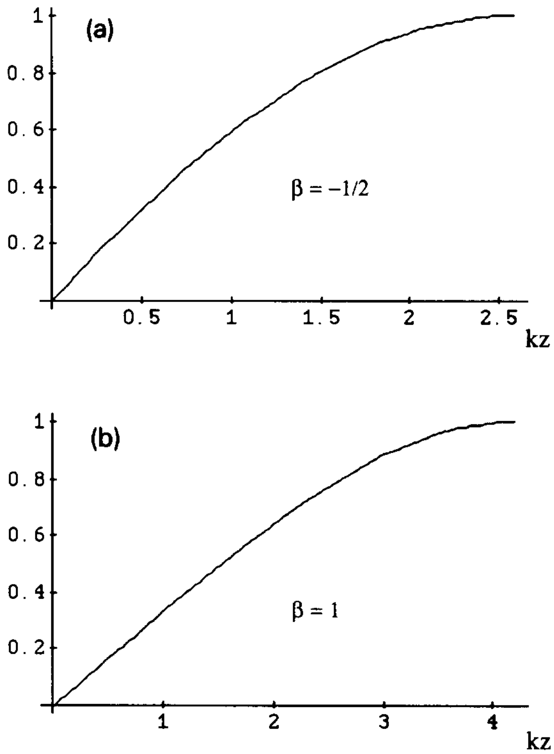

Fig. 1. The solutions for $(A/u_z)^{1/(\beta+2)}R$. (a) $\beta = -\frac{1}{2}$, and (b) $\beta = 1$. The values of k on the abcissa are scaled such that the period is $[4/(\beta+2)]\ \mathscr{B}[1/(\beta+2), \frac{1}{2}]$.

For later use, I also identify an analogous wave number k, defined such that the period is unity (not 2π):

$$k = (A/u_z)^{1/(\beta+2)} \frac{(\beta+2)u_z^{1/2}}{4\mathscr{B}[1/(\beta+2), \frac{1}{2}]} . \tag{10}$$

Turning to the temporal behaviour, eq. (5b) can be solved in the same manner as (5a) through multiplication by $\partial_t T$ and integration, which yields

$$(\partial_t T)^2 + A_1|T|^{(\beta+2)/(\beta+1)} = u_t , \tag{11}$$

where u_t is a constant of integration and $A_1 \equiv A\nu_0^2(\beta+1)$. Similarly to eq. (6), for all $-1 < \beta$ (11) describes a non-dissipative oscillatory behaviour in a generally non-quadratic potential well. The amplitude of T can be found by considering the instant when $\partial_t T = 0$, which gives

$$a_T = (u_t/A_1)^{(\beta+1)/(\beta+2)} . \tag{12}$$

From (4), (7) and (12) we can now deduce the amplitude of the electric field:

$$E_0 = a_R a_T^{1/(\beta+1)} = \left[\left(\frac{\beta+2}{2K} \right)^2 \frac{u_z u_t}{\nu_0^2(\beta+1)} \right]^{1/(\beta+2)} . \tag{13}$$

Exactly as above, eq. (11) can be inverted to solve for the time t

$$(A_1/u_t)^{(\beta+1)/(\beta+2)} \frac{\beta+2}{\beta+1} u_t(t-t_0) = \pm \mathscr{B}_\eta \left(\frac{\beta+1}{\beta+2}, \frac{1}{2} \right) , \tag{14}$$

where $\eta = (A_1/u_1)|T|^{(\beta+1)/(\beta+2)}$, and t_0 is some initial time. When T is expressed in terms of t, relation (14) yields the expected oscillatory behaviour with a period of $[4(\beta+1)/(\beta+2)] \times \mathscr{B}((\beta+1)/(\beta+2), \frac{1}{2})$. This period also reduces to 2π when $\beta = 0$, as expected. The frequency of the oscillation can be found from (14):

$$\omega = (A_1/u_t)^{(\beta+1)/(\beta+2)} u_t^{1/2} / \mathscr{B} \left(\frac{\beta+1}{\beta+2}, \frac{1}{2} \right) . \tag{15}$$

In the linear case the ratio ω/k is significant as it gives the dispersion relation and the phase velocity, therefore it is of interest to consider it for our case. Using eqs. (10) and (15) yields

$$\omega/k = E_0^{-\beta/2} \sqrt{\beta+1}\, \nu_0 \mathscr{B} \left(\frac{1}{\beta+2}, \frac{1}{2} \right) \Big/ \mathscr{B} \left(\frac{\beta+1}{\beta+2}, \frac{1}{2} \right) , \tag{16}$$

which reduces to the usual ν_0 in the linear case as it should. Expression (16) allows for a spectacular interpretation: Since E_0 and β depend neither on time nor on spatial coordinates ω is exactly *linear* in k, with the linearity coefficient depending on the nonlinearity parameter β and the amplitude of the externally applied field E_0.

The first implication of this intriguing result is that despite being strongly nonlinear, a non-dissipative medium can sustain a persistent standing wave. Namely, pumping a wave of frequency ω at $z = 0$ onto a medium, confined between $z = 0$ and $z = W < \infty$, results in a standing wave. This wave will assume definite periodicities that are integer multiples of the basic period, exactly as in the linear case. But unlike the linear case a simple linear combination of these solutions is not a solution because the superposition principle does not apply. The second ramification is practical and relates to the intensity dependence. Relation (16) shows that increasing the amplitude E_0 of the incident wave at $z = 0$ decreases (increases) the phase velocity for $\beta > 0$ $(-1 < \beta < 0)$. Thus one can modulate the wave number k of such a standing wave by varying the *amplitude* E_0 rather than the frequency ω of the source. While it is not unusual to find that nonlinearity couples frequency and amplitude, it is not every day that one can find an *exact* relation between the two, rather than a truncation of some expansion in a small parameter.

So the electric field inside the nonlinear medium oscillates in space and time, with the oscillating functional form being the inverse of the incomplete beta function with respect to its index. Regarding boundary conditions, we note that for the calculation to hold first the incident wave at $z = z_0$ has to be prearranged such that it follows the same form as $R(z = z_0)\, T(t = t_0)^{1/(\beta+1)}$. If this is not taken care of one may end up with a mismatch at the boundary leading to a possible dispersion at $z = 0^+$. Since the superposition principle cannot be applied in our medium, such a situation complicates the analysis. Secondly, the reflected amplitude at z_0 should equal the incident.

Next let us consider the behaviour of the magnetic field $\boldsymbol{H}(z, t)$ within the medium. Rewriting eqs. (2) in terms of the functions R and

T, integrating and using (5), (6) and (11), H is found explicitly:

$$H = \pm \frac{\varepsilon}{cK}\, \partial_z R\, \partial_t T = \pm \frac{\varepsilon (u_z u_t)^{1/2}}{cK} \sqrt{[1-(A/u_z)|R|^{\beta+2}]} \times \sqrt{[1-(A_1/u_t)|T|^{(\beta+2)/(\beta+1)}]}\,. \tag{17}$$

The expressions within the square roots are the canonical wave solutions to the energetic eqs. (6) and (11), which vary with z and t between 0 and 1. The amplitude of H is then the prefactor, which can be identified, using (13), as

$$H_0 = \frac{2}{\beta+2}\sqrt{\frac{\varepsilon(\beta+1)}{\mu}}\, E_0^{(\beta+2)/2}\,. \tag{18}$$

Having found the explicit forms of the electric and the magnetic fields, I now discuss the energy stored in the field and its flow in the nonlinear medium. Consider Poynting's vector

$$\boldsymbol{S} = (c/4\pi)\, \boldsymbol{E} \times \boldsymbol{H}\,. \tag{19}$$

The divergence of $\boldsymbol{S}$ can be easily calculated in a general medium [7]

$$\operatorname{div} \boldsymbol{S} + (\boldsymbol{H}\cdot\partial_t \boldsymbol{B} + \boldsymbol{E}\cdot\partial_t \boldsymbol{D})/4\pi = 0\,,$$

and in the case of the nonlinear constitutive relations (1), can be written as

$$\operatorname{div} \boldsymbol{S} + \partial_t U = 0\,, \tag{20}$$

where

$$U = \frac{1}{4\pi}\left(\frac{\beta+1}{\beta+2}\, \varepsilon |E|^{\beta+2} + \tfrac{1}{2}\mu |H|^2\right) \tag{21}$$

is *exactly the energy density* in the system. It should be stressed that although eq. (20) holds for any nonlinear system, U need not, and generally does not, coincide with the energy density in the system. Thus within a period of oscillation the energy density is exchanged between the magnetic and electric fields.

It should be emphasized that by assuming a separable solution, the above discussion focused implicitly on a standing, rather than a propagating, wave. So let us now turn to consider the propagation of a signal in such a medium. In the linear case the fields E and H can be written as functions of the reduced variable $x = z \pm \nu_0 t$, which shows that a signal of a well defined unique frequency will propagate at the speed of light ν_0 both forward and backward in the corresponding medium. The question is can one identify in our case a quantity analogous to ν_0. Let us assume that there exists such a velocity ν_β. The field E can then be written as a function of the reduced variable $\xi = (t - t_0) - (z - z_0)/\nu_\beta$ (to simplify the notation only forward propagation is considered). The first partial derivatives of E can be rewritten as

$$\partial_z E = \frac{E'}{\nu_\beta[1 + (z - z_0)\, \mathrm{d}_E(1/\nu_\beta)E']}\,,$$
$$\partial_t E = \frac{E'}{1 + (z - z_0)\, \mathrm{d}_E(1/\nu_\beta)E'}\,, \tag{22}$$

where $\mathrm{d}_E = \mathrm{d}/\mathrm{d}E$ represents derivative with respect to the explicit dependence on E and where E' is the derivative of E with respect to the reduced variable ξ. Assuming that $[1 + (z - z_0)\, \mathrm{d}_E(1/\nu_\beta)\, E']$ does not vanish we have

$$\partial_z E = -(1/\nu_\beta)\, \partial_t E\,,$$

which, combined with the identity $\partial_z E/\partial_t E = -1(\partial z/\partial t)_E$, simply states that

$$(\partial z/\partial t)_E = \nu_\beta\,. \tag{23}$$

Namely, ν_β is the velocity along the trajectories of constant field E in the z–t plane. I now claim that

$$\nu_\beta = \nu_0 |E|^{-\beta/2}/\sqrt{\beta+1}\,, \tag{24}$$

and proceed to prove it by showing that this expression solves Maxwell's equations. Noting that $1/\nu_\beta^2 = (\mu/c^2)\, \mathrm{d}_E D$ and using (2a) one has

$$\partial_z H = -(c/\mu)(1/\nu_\beta^2)\, \partial_t E\,. \tag{25}$$

Using (23) in (25) and changing variables further yields

$$H = (c/\mu) \int \mathrm{d}E/\nu_\beta\,,$$

which can be checked by differentiation with respect to time and comparing with (2b). Explicitly, this relation yields for the magnitude of $\boldsymbol{H}$

$$H = \frac{2\sqrt{(\beta+1)\varepsilon/u}}{\beta+2}\, |E|^{\beta/2} E\,, \tag{26}$$

which coincides with (17) and (18) up to a phase shift. Expressions (26) also shows that for a propagating signal, H is *in phase* with E (rather than in antiphase as for a standing wave), exactly as in the linear case.

Further, this calculation shows that *any function* of ξ solves Maxwell's equations for E and H. This generalizes the linear case result, where any function of $z - \nu_0 t$ is a solution, depending on the initial conditions.

So by writing E in terms of ξ the problem is reduced to being described by one variable rather than two. This reduction may seem cumbersome due to the dependence of ξ on E through ν_β, but it is still useful as it provides insight when analysing the stability of the form of a propagating signal in such a medium. This issue will not be addressed here [5], but I will only remark that a signal propagating with a velocity that follows (24) may evolve into a frontal or rear shock-wave-like form [8].

The mean velocity of propagation $\langle \nu_\beta \rangle$, can be found in two ways: One is by averaging (25) directly over z and t, which gives $\langle \nu_\beta \rangle = \nu_0 \langle E^{-\beta/2} \rangle / (\beta+1)^{1/2} \sim E_0^{-\beta/2}$. Another way is to consider the propagation of the energy flux through the media. The general relation (20) has the form of a continuity equation and, when averaged over z and t, constitutes the conservation of energy in the system. The velocity of propagation is then simply $\langle S \rangle / \langle U \rangle$. Using $\boldsymbol{S}$ and U for the previous case of separable solution yields

$$\langle \nu_\beta \rangle = \frac{2(\beta+2)}{(\beta+4)(\beta+1)^{1/2}}\, \nu_0\, \frac{\langle E^{\beta/2+1} \rangle}{\langle E^{\beta+2} \rangle} \sim E_0^{-\beta/2}\,,$$

which varies with the same power of the field amplitude as in the first method. As expected, this power vanishes in the linear case, leading to the familiar field-independent constant velocity.

Thus, ν_β indeed represents a local and instantaneous velocity of propagation of the solution in the medium and it varies with the spatial and temporal coordinates, tracing the variation of the field. Information can be transmitted by this solution, e.g., by introducing a perturbation at some t and z. Although, unlike in the linear case, the propagation of a general perturbation is difficult to analyse, this perturbation will distort the field, and such a distortion can be detected at another location at a later time. This information propagates with the average velocity, and hence the instantaneous velocity is not the most significant quantity to the passage of information over extended distances.

To conclude, I have analysed the electromagnetic response of a strongly nonlinear dielectric medium. I have presented exact and explicit wave solutions to Maxwell's equations. The frequency of oscillation ω has been shown to be linear in the analog of the wave number k, which indicates that such a medium can support a standing wave. The proportionality coefficient ω/k has been shown to vary as a power of the intensity of the incident wave, which allows the wave length inside the medium to be modulated by changing the intensity of the source wave, rather than varying its frequency. The energy flux and Poynting's vector have been solved for and discussed for these solutions. The velocity derivative $\partial z/\partial t$ along trajectories of constant field in the z–t plane has been found to be a power of the field intensity for any general solution. In particular, when the field oscillates the velocity traces this oscillation and consequently the characteristic lines of constant field may also oscillate periodically around the straight line. Some points remain unclear in the propagating case regarding the stable form of the signal. The simplicity in the linear case stems from the easy decomposition of the plane wave in $x = z - v_0 t$ into a sum of two separable periodic functions in kz and ωt. Such a decomposition is not available in the present nonlinear problem, and this question is currently under study.

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