Nonequilibrium Brittle Fracture Propagation: Steady State, Oscillations, and Intermittency

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A minimal model is constructed for two-dimensional fracture propagation. The heterogeneous process zone is presumed to suppress stress relaxation rate, leading to non-quasistatic behavior. Using the Yoffe solution, I construct and solve a dynamical equation for the tip stress. I dicuss a generic tip-velocity response to local stress and find that noise-free propagation is either at steady state or oscillatory, depening only on one material parameter. Noise gives rise to intermittency and quasiperiodicity. The theory explains the velocity oscillations and the complicated behavior seen in polymeric and amorphous brittle materials. I suggest experimental verifications and new connections between velocity measurements and material properties. [S0031-9007(96)00162-7]

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The dynamics of cracks propagating in amorphous brittle media focused extensive study since the forties, mostly through quasistatic approaches and energetic arguments. In spite of recently renewed interest there are several fundamental issues that seem difficult to resolve in any simple way. For example, the limiting crack velocity, predicted to be the Rayleigh wave speed (RWS) in the bulk [1], is observed to be only about half of that; the mechanism for crack initiation and arrest is poorly understood; and the occurrence of velocity oscillations [2] is still a puzzle. At the heart of the problem is the fact that the system's behavior depends on the length scale. While it is evident that the atomistic behavior differs from the continuous, it is this author's opinion that even on the continuum scale the physics near the tip is distinct from that far away and therefore should be treated differently. This may explain an apparent discrepancy: On the one hand, since the bulk shear wave speed (SWS) is higher than the crack velocity, it is clear that far from the crack quasistatic arguments should work well because the field relaxes to its static form, $\sigma_{\alpha\beta} = Kf_{\alpha\beta}(\theta)/\sqrt{2\pi r}$, sufficiently fast. Here $\sigma_{\alpha\beta}$ is the stress tensor, r is the distance from the crack tip, $f_{\alpha\beta}$ depends only on the azimuthal angle θ, and K is the (time-dependent) stress intensity factor. On the other hand, the inability of quasistatic theories to account for the above phenomena suggests that much of the behavior is determined by the *local dynamics at the tip* and hence that the propagation is a far-from-equilibrium process, indescribable by approaches that appeal to energy balancing. In the two-scale picture the nonequilibrium dynamics act, in effect, to dress the tip singularity as seen from afar. The matching of the near and far fields at the crossover scale then yields the far-away behavior of K. A reasonable guess would be that the crossover scale is of the order of the size of the process zone (PZ) in front of the propagating crack. While the far quasistatic field is well understood within linear elasticity, there is little understanding of the short-range physics, although a few phenomenological dynamic equations have been advanced [3,4] to explain the limiting tip velocity.

It has been conjectured [5,6] that the reason for the complicated short-range behavior is the heterogeneous and fluid structure of the PZ. This conjecture may be supported by observations of extremely slow relaxation rates of the stress at the tip after arrest [7], rates that are an order of magnitude below expectation had the relaxation taken place at the bulk speed of sound. This implies that the waves that re-establish the stress field in the PZ travel at a speed, c, that is *much lower* than the bulk SWS, probably due to scattering from microvoids.

The model proposed here concerns the short-range dynamics and takes on board several ingredients: the low value of c in the PZ, the occurrence of different stresses for crack initiation and arrest [5,8], and, based on existing observations, an assumed velocity response to the tip stress. These suffice to construct and solve a dynamic equation. The explicit form of the velocity-stress relation is not required for most of the results obtained here, only its qualitative behavior. The model leads to either a steady-state propagation at a limiting velocity or an oscillatory behavior, with the selection between the two modes dependent on the location of the suppressed speed c on the velocity response function. Introduction of noise due to microvoid distribution is shown to give rise to an intermittent propagation that can turn into a quasiperiodic behavior.

Consider a line crack (not necessarily straight) in a two-dimensional material. The PZ in front of the tip is modeled as an effective continuous medium with a reduced SWS, c. The dependence of the crack dynamics on material properties enters through a velocity response function, $v(\sigma)$ [9], where σ is the local stress at the tip in the forward direction. As the crack propagates, the field near the tip adjusts to the changing boundary at a rate that corresponds to c. Observations that steady-state propagation is at about half the bulk RWS, combined with the fact that c is much lower than the bulk (homogeneous) RWS, implies that the tip velocity can *momentarily exceed* the local value of c. This is a basic assumption in what follows. I comment that this does not violate the energy balance which holds for scales away from the PZ, because near the tip the

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dynamic response $v(\sigma)$ is swifter than the global energy equilibration process. The measured behavior of $v(\sigma)$ is hysteric with two material-dependent thresholds: σ_h, above which propagation initiates, and $\sigma_l < \sigma_h$, to which the stress has to drop for the crack to arrest [5,8]. For $\sigma > \sigma_h$ the velocity is also known to increase very slowly with stress [10,11]. Figure 1 shows a qualitative form of $v(\sigma)$ that is consistent with experimental observations. This local nonmonotonic respond differs from that in [4] which depends on energy equilibration far from the tip. Its locality allows one to find the dynamics without further assumptions. A local two-branch velocity can be derived from atomistic models [12].

To derive the equation of motion of the tip, let us start from the Yoffe solution for the forward field of a propagating crack of length a [13],

$$\sigma = \sigma_\infty[(\zeta + a)/\sqrt{\zeta(\zeta + 2a)}], \tag{1}$$

where ζ is the distance from the tip and σ_∞ is the tensile stress applied perpendicular to the propagation axis far away from the crack. In what follows, the stress is measured in units of σ_∞, and $\sigma \to \sigma/\sigma_\infty$ is dimensionless and >1. This solution assumes that the singularity of the field is always at the tip, which is consistent with a quasistatic picture. Consider, however, a situation wherein the dynamic response constrains the tip to overtake the density waves that adjust the field. In this situation the singularity in the stress field *does not* coincide with the location of the tip, and the tip's stress drops to below the static value. The difference between the static and dynamic stresses at the tip depends on the tip's velocity $v = dl/dt$ and the propagation history. The dynamic stress is found from (1) by putting $\zeta = (l - ct)\Phi(l - ct)$, where l is the tip's position and the step function Θ ensures that, when the shear wave catches up, the tip stress stays at the static value. When $\Theta = 0$ the tip stress diverges as expected and traditional quasistatic solutions apply [14]. Focusing on nonquasistatic propagation, I assume $\Theta = 1$ during the entire growth. When Θ alternates between 0 and 1 one simply pieces the solutions together. Taking the time

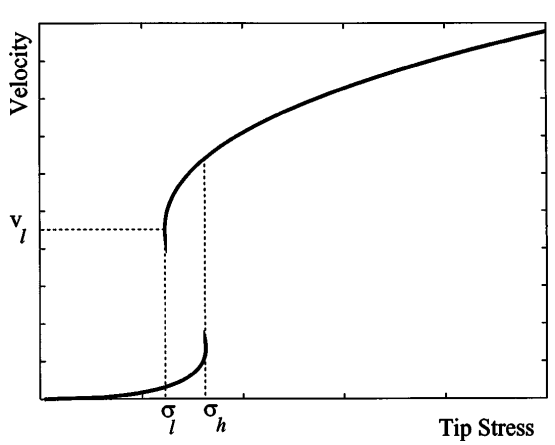

FIG. 1. A generic plot of $v(\sigma)$.

derivative of (1), we have

$$\dot{\sigma} = -\dot{\zeta}a^2/[\zeta(\zeta + 2a)]^{3/2}. \tag{2}$$

Using (1), we can invert relation (2):

$$\dot{\zeta}/a = -\dot{\sigma}/(\sigma^2 - 1)^{3/2}. \tag{3}$$

We define $\dot{\zeta} = (v - c)$ as $\dot{\zeta} \equiv cu$, where u is a reduced velocity. Upon substitution in (3), we can readily solve

$$t - t_0 = -\frac{a}{c}\int_{\sigma(t_0)}^{\sigma(t)} \frac{ds}{(s^2 - 1)^{3/2}u(s)}. \tag{4}$$

The kinetics are thus determined by the response function through the stress dependence of $u(\sigma)$. Relation (4) is the bare result of this Letter. It is an exact derivation from the Yoffe solution. It gives the general time dependence of the stress at the crack tip. Once the stress history is found from this relation, one substitutes it in $v(\sigma)$ to obtain the velocity history. We now proceed to analyze the consequences of this result, assuming the qualitative response shown in Fig. 1. It is convenient to classify the behavior in terms of the ratio $\lambda = c/v_l$. The reason is that, as is shown below, the mode of propagation depends only on this ratio.

$\lambda > 1$.—The point $(\sigma(c), c)$ is on the upper branch of $v(\sigma)$. Suppose that initially $\sigma < \sigma_l$. The velocity is momentarily zero (or very low), and the tip stress builds up to σ_h. At this stage the system "jumps" to the upper branch and fast motion ensues. From relation (4) we see that for $\sigma_h > \sigma(c)$ [$<\sigma(c)$] the stress will decrease [increase] until σ converges to $\sigma(c)$, whereafter the tip propagates at a velocity c and a fixed distance ahead of the density waves. Thus $(\sigma(c), c)$ is a *fixed point* of the equation of motion. The behavior at the vicinity of this point can be found by linearization of relation (4):

$$|\sigma - \sigma(c)| \approx Ce^{-\gamma t},$$
$$\gamma \equiv \frac{[\sigma(c)^2 - 1]^{3/2}}{a}\left(\frac{dv}{d\sigma}\right)_{\sigma(c)}. \tag{5}$$

Since $v(\sigma)$ near $\sigma(c)$ is smooth and positive γ is regular and positive and the fixed point is *stable*, namely, steady-state propagation at a limiting velocity c is a stable fixed point of the dynamics. A typical such history of v is shown in Fig. 2. An interesting implication of this result is that the experimentally observed limiting crack velocities give, in fact, the value of c and hence the local stress relaxation rate. This suggests a check of this model by comparing the limiting velocity to the speed of sound in the PZ. It is intriguing to note that even in the absence of a global energy balance criterion the crack velocity converges to the SWS, albeit the local value, c. Observations that v increases very slowly with K [15] in this regime indicate a small value of $dv/d\sigma$ along the upper branch. In view of the present analysis, this agrees with the reported velocity behavior immediately after crack initiation [5,8,10]. Another check of this picture can be suggested: In some experiments a drop in the

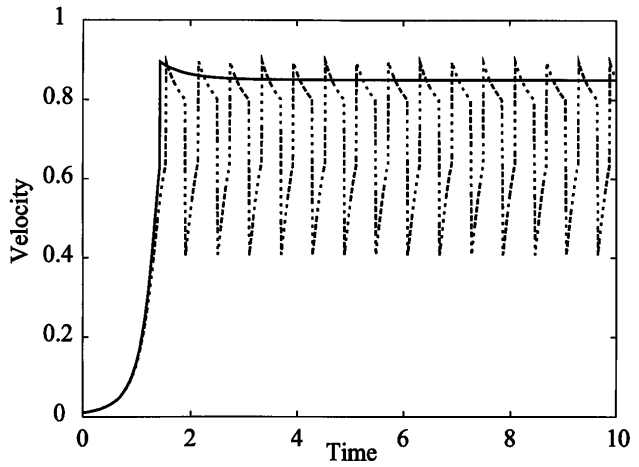

FIG. 2. Typical velocity histories in the steady-state (solid) and periodic (dashed) regimes.

stress has been measured after crack initiation [16]. This suggests that in those systems $\sigma(c) < \sigma_h$, a conclusion that can be checked by independent methods as a test of this analysis.

$\lambda < 1$.—To analyze this case, let us assume again that initially the tip stress is lower than σ_l. From relation (4) the stress will increase until it reaches σ_h, whereupon the crack will start propagating as for $\lambda > 1$. The velocity and the stress will then gradually decrease. Since $c < v_l$ is not a point on the upper branch, the system cannot settle into a steady state as before and at σ_l it flips back to the lower branch. There the crack halts momentarily, the stress at the tip builds up again to σ_h, and the cycle repeats itself. This is a *relaxation cycle* whose period is found from (4):

$$\tau = \frac{a}{c}\int_{\sigma_l}^{\sigma_h} \frac{1/u_{ub}(s) - 1/u_{lb}(s)}{(s^2 - 1)^{3/2}}\, ds\,, \tag{6}$$

where $u_{ub} > 0$ and $u_{lb} < 0$ are, respectively, the values of u along the upper and lower branches. When the velocity vanishes along the lower branch $u_{lb} = -1$. A typical velocity history in this case is also shown in Fig. 2.

$\lambda = 1$.—This marginal case is sensitive to the value of $\partial v/\partial\sigma$ at σ_l. If the derivative is regular, one can easily see that the analysis is the same as for $\lambda > 1$. The only difference is that σ can only approach $\sigma(c) = \sigma_l$ from above because for $\sigma < \sigma_l$ the only motion is up the lower branch. If $\partial v/\partial\sigma$ diverges at σ_l, the behavior depends on the detailed form of the divergence. For illustration, consider the form

$$v = v_l \exp\left[\alpha\left(\frac{\sigma}{\sigma_l} - 1\right)^{\nu}\right] \rightarrow u \sim \text{const} + (\sigma - \sigma_l)^{\nu}, \tag{7}$$

with $0 < \nu < 1$. The behavior near the fixed point is found by using (7) in Eq. (4),

$$\sigma - \sigma(c) \sim (\tau_0 - t)^{1/(1-\nu)}, \quad u \sim (\tau_0 - t)^{\nu/(1-\nu)}, \tag{8}$$

where $\tau_0 > t$ is a constant. Now the propagation rate converges to c as a *power law* rather than exponentially. It should be noted that this propagation mode is sensitive to small fluctuations that can easily flip the system to the lower branch. A small noise in this case will give rise to a quasiperiodic motion similar to that discussed below.

Intermittency and quasiperiodicity.—Since the PZ is inhomogeneous, one expects fluctuations in the local properties which may well go beyond the effective medium assumption. For simplicity, let us restrict the discussion only to fluctuations in the tip stress during propagation, $\sigma = \sigma_0(t) - \eta(t)$, where σ_0 is the stress in the absence of noise. This corresponds to a situation where the crack encounters microvoids of varying sizes along its path. Upon association of a microvoid to the crack, the tip stress drops momentarily with the drop depending on the microvoid's size. The fluctuations in microvoid sizes give rise then to noise in the tip stress. A fluctuation during steady-state propagation that reduces σ by more than $\delta = \sigma(c) - \sigma_l$ (see Fig. 1) flips the system to the lower branch. The system then has to go through the process of stress increase to σ_h, jump to the upper branch, and converge to $\sigma(c)$ again. The time that this process takes depends on the original fluctuation, $\eta > \delta$, and can be found by applying (4) to the motion along the two branches:

$$T(\eta) = \frac{a}{c}\left[\int_{\sigma(c)}^{\sigma_h} \frac{u_{ub}(s)^{-1} ds}{(s^2 - 1)^{3/2}} - \int_{\sigma(c)-\eta}^{\sigma_h} \frac{u_{lb}(s)^{-1} ds}{(s^2 - 1)^{3/2}}\right]. \tag{9}$$

It is the occurrence frequency of the flips between the branches which determines to a large extent the observable behavior. This frequency depends on both the noise characteristics and the value of δ. The stochastic velocity behavior can be obtained from the statistics of η by using relation (9). For example, the probability density of T, $P(T)$, can be found from that of η, $P_0(\eta)$, by inverting relation (9) to obtain $\eta(T)$ and substituting in

$$P(T) = P_0(\eta(T))d\eta/dT\,. \tag{10}$$

From (9) one can also find the effects of various forms of the noise temporal correlations $\langle\eta(t)\eta(t')\rangle$ on the velocity history. A detailed analysis of the statistics, including the explicit dependence on the distribution of microvoid sizes, will be reported shortly. Here I only point out a few intriguing consequences for $\lambda > 1$ and $\sigma(c) < \sigma_h$: First, a low occurrence frequency (i.e., $< 1/\tau$) of $\eta > \delta$ leads to an *intermittent* behavior, wherein the tip is "knocked" occasionally from the steady state and then returns to it only to be knocked out of it again at a later time. A plot of such a history is shown in Fig. 3. Second, a very high occurrence frequency of fluctuations of size $\eta > \sigma_h - \sigma_l$ gives rise to a quasiperiodic behavior as follows: As the tip stress builds up along the lower branch to σ_h the system flips to the upper branch. A fluctuation then immediately knocks the system back to the lower branch, not allowing it to settle into the steady state. The mean period will then be close to the time spent

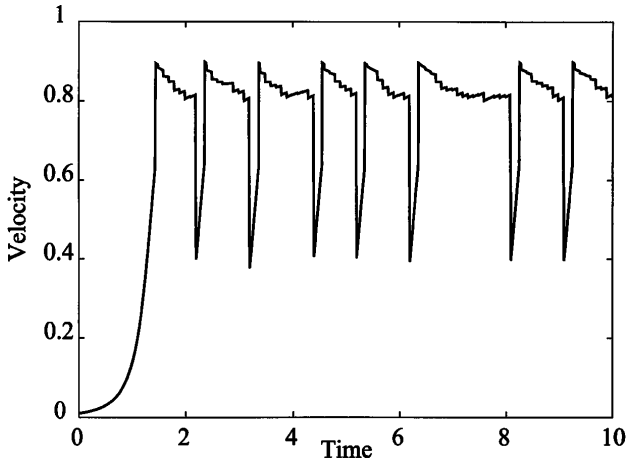

Fig. 3. A typical velocity history in the intermittent regime.

on the lower branch, namely, $\approx(a/c)[(1-1/\sigma_l^2)^{-1/2} - (1-1/\sigma_h^2)^{-1/2}]$. Thus the microvoid size distribution determines the observable behavior by governing the statistics of η. Since this distribution also plays a major role in determining the roughness of the fracture surfaces, my analysis can rigorously link roughness measurements to the velocity history.

To summarize, a minimal theoretical model has been proposed to explain the rich behavior observed in crack propagation in amorphous and polymeric materials. The theory is a direct consequence of the observations of slow relaxation rates of the tip stress, the occurrence of different initiation and arrest stresses, and the deduced qualitative form of $v(\sigma)$. The mode of propagation has been found to depend only on one material parameter, $\lambda = c/v_l$. For $\lambda > 1$ the propagation speed saturates to a limiting value, while for $\lambda < 1$ it oscillates periodically. Noise gives rise to a spectrum of behavior ranging from intermittent to quasiperiodic propagation. The model explains naturally recent observations of oscillations in polymeric materials, and measurements for its validation have been suggested. It predicts that the low-noise steady-state growth rate is exactly c, the wave speed in the PZ, and should be possible to test experimentally. Low λ is expected to correspond to high disorder and vice versa. So, by manipulating the disorder, one may tune λ. I should remark that the effective continuum approximation of the PZ probably breaks down for too broad a distribution of microvoid sizes, and a statistical treatment is more adequate. E.g., for propagation velocities of order $c \sim 500$ m/s and given the fact that currently observations are limited to times of μsec and higher, the effective continuum assumption should hold when microvoids are smaller than 500 μm. Microvoids do not usually reach such sizes in polymeric materials, and therefore this model should do a good job explaining experimental observations [2] in these systems. A complementary statistical analysis for broad microvoid distributions is currently under way and will be reported shortly. Finally, many ramifications of this model remain to be explored: the effects of noise correlations $\langle \eta(t)\eta(t')\rangle$ on the dynamics, the effects of realistic distribution of void sizes, and the implications of the statistics and velocity history on the surface roughness, to name a few.

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